Non-negative Weak Majority Roman Domination In Graphs

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Abstract

A Non-negative Weak Majority Roman Dominating Function (NNWMRDF) on a graph G=(V,E) is a function $f : V \to \{-1, +1, 2\}$ satisfying the condition that the sum of its function values over at least half the closed neighborhood is at least zero. The weight of a NNWMRDF is the sum of its function values over all vertices. The *Non-negative Weak Majority Roman Domination Number* of a graph G, denoted by $\gamma^{NN}_{wmr}(G)$, is defined as $\gamma^{NN}_{wmr}(G) = min\{w(f) \mid f\ is\ a\ NNWMRDF\ of\ G\}$. In this paper, we initiate the study of Non-negative Weak Majority Roman Domination In Graphs.

Keywords: Non-negative Weak Majority Roman domination, Non-negative Weak Majority Roman Number.

Mathematics Subject Classification: 05C15, 05C69.

1 Introduction

By a graph $G = (V, E)$, we mean a finite, non-trivial, connected, and undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartand and Lesniak [1].

The study of domination is one of the fastest growing areas within graph theory. A subset D of vertices is said to be a *dominating set* of G if every vertex in V either belongs to D or is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Survey of several advanced topics on domination is given in the book edited by Haynes et al. [2].

For a real valued function $f : V \to R$ on V,*weight of f* is defined to be $w(f) = \sum_{v \in V} f(v)$ and also for a subset $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. Therefore $w(f) = f(V)$. Further, for a vertex $v \in V$, let $f[v] = f(N[v])$ for notation convenience. A function $f : V \to \{-1, +1\}$ is called a *majority dominating function* if $f[v] \geq 1$ for at least half of the vertices in G. The *majority domination number* of G is denoted by $\gamma_{maj}(G)$ and is defined as $\gamma_{maj}(G) = min\,\{w(f) \mid f\ is\ a\ majority\ dominating\ function\ of\ G\}$. Majority domination was first introduced by Broere et al. in [3] and further studied in [4, 5].

A function $f : V \to \{-1, +1\}$ is called a *Non-negative signed dominating function* if $f(N(v)) \geq 0$ for all vertices in graph G. The *Non-negative signed domination number* of G, is defined as $\gamma_S^{Nt}(G) = min\,\{w(f) \mid f\ is\ a\ NSDF\ of\ G\}$. The concept of non-negative signed domination of a graph was introduced in [7].

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A weak signed Roman dominating function (WSRDF) of a graph G with vertex set $V(G)$ is defined as a function f : V (G) $\to \{-1, +1, 2\}$ having the property that $f(N[v]) \geq 1$ for all $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. The weight of a WSRDF is the sum of its function values over all vertices. The weak signed Roman domination number of G, denoted by $\gamma_{wsR}(G)$, is the minimum weight of a WSRDF in G. Weak signed Roman domination number was introduced by L.Volkmann in [6]

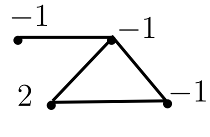
In this paper, we initiate the study of Non-negative Weak Majority Roman Domination in graphs.

2 Definition

Definition 2.0.1. A Non-negative Weak Majority Roman Dominating Function (NNWMRDF) on a graph G=(V,E) is a function $f : V \to \{-1, +1, 2\}$ satisfying the condition that the sum of its function values over at least half the closed neighborhood is at least zero. The weight of a NNWMRDF is the sum of its function values over all vertices. The *Non-negative Weak Majority Roman Domination Number* of a graph G, denoted by $\gamma^{NN}_{WMR}(G)$, is defined as $\gamma^{NN}_{WMR}(G) = min\,\{w(f) \mid f\ is\ a\ NNWMRDF\ of\ G\}$.

Remark 2.0.2. For every graph G, $\gamma^{NN}_{WMR}(G) \leq \gamma_{wsR}(G)$.

Example 2.0.3. Now consider the graph G as follows



By the way of assigning −1,+1 and 2 to the vertices of G, it is easy to observe that $\gamma^{NN}_{WMR}(G) = -1$.

Remark 2.0.4. Let us follow throughout the paper the following terminologies. If f is a weak majority roman dominating function of a graph G, then we define the sets $V_{f,1}$, $V_{f,-1}$, $V_{f,2}$ and N_f as follows.

(i) $V_{f,1}(G) = \{v \in V(G) : f(v) = 1\}$

(ii) $V_{f,-1}(G) = \{v \in V(G) : f(v) = -1\}$

(iii) $V_{f,2}(G) = \{v \in V(G) : f(v) = 2\}$

(iv) $N_f(G) = \{v \in V(G) : f[v] \geq 0\}$

Remark 2.0.5. If f is any weak majority Roman dominating function of a graph G of order n, then $f[v] \geq 0$ for at least half of the vertices of G. Further, it is obvious that $|V_{f,1}| + |V_{f,-1}| + |V_{f,2}| = n$ and $\gamma^{NN}_{WMR}(G) \leq |V_{f,1}| - |V_{f,-1}| + 2\,|V_{f,2}|$.

3 Common Classes Of Graphs

Theorem 3.0.6. *For any path P_n on $n \geq 2$ vertices,*

$$\gamma^{NN}_{WMR}(P_n) = \begin{cases} 3\left\lceil \frac{n}{6} \right\rceil - n - 1 & \text{if } \left\lceil \frac{n}{2} \right\rceil \equiv 1\,(mod\ 3)\,, \\ 3\left\lceil \frac{n}{6} \right\rceil - n & \text{otherwise.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, ..., v_n)$ and let f be a NNWMRDF of P_n with $\gamma^{NN}_{wmr}(P_n) = f(V)$. Since $\deg v_i = 2$, for all $(2 \leq i \leq n-1)$, for a vertex $v_i \in V_{f,2}$, three vertices belongs to N_f. Since $|N_f| \geq \left\lceil \frac{n}{2} \right\rceil$, we have three cases.

Case 1. $\left\lceil \frac{n}{2} \right\rceil \equiv 0\,(mod\ 3)$

Then $|V_{f,2}| \geq \left\lceil \frac{n}{6} \right\rceil$ and $|V_{f,-1}| \leq n - \left\lceil \frac{n}{6} \right\rceil$. Hence $\gamma^{NN}_{WMR}(P_n) \geq 3\left\lceil \frac{n}{6} \right\rceil - n$.

Case 2. $\left\lceil \frac{n}{2} \right\rceil \equiv 1\,(mod\ 3)$

That is, $\lceil \frac{n}{2} \rceil - 1 \equiv 0\,(mod\ 3)$. Then $|V_{f,2}| \geq \lceil \frac{n}{6} \rceil - 1$ and $|N_f| \geq \lceil \frac{n}{2} \rceil - 1$. Hence $|V_{f,1}| \geq 1$ and $|V_{f,-1}| \leq n - \lceil \frac{n}{6} \rceil$. Therefore $\gamma^{NN}_{WMR}(P_n) \geq 3 \lceil \frac{n}{6} \rceil - n - 1$.

Case 3. $\lceil \frac{n}{2} \rceil \equiv 2\,(mod\ 3)$

That is, $\lceil \frac{n}{2} \rceil - 2 \equiv 0\,(mod\ 3)$. Then $|V_{f,2}| \geq \lceil \frac{n}{6} \rceil - 1$ and $|N_f| \geq \lceil \frac{n}{2} \rceil - 2$. If the two pendant vertices belongs to N_f, then $|V_{f,1}| \geq 2$. Otherwise $|V_{f,2}| \geq \lceil \frac{n}{6} \rceil$ and $|V_{f,-1}| \leq n - \lceil \frac{n}{6} \rceil$. Therefore $\gamma^{NN}_{WMR}(P_n) \geq 3 \lceil \frac{n}{6} \rceil - n$

On the other hand, define the function $g : V \to \{-1, +1, 2\}$ by

(i) $n \equiv x\,(mod\ 6)$, where $x \in \{0, 4, 5\}$

$$g(v_i) = \begin{cases} 2 & \text{if } i = 2 \text{ and } \text{ i} \equiv 2\,(mod\ 3), (1 \leq i \leq \lfloor \frac{n}{2} \rfloor) \\ -1 & \text{otherwise} \end{cases}$$

(ii) $n \equiv 3\,(mod\ 6)$

$$g(v_i) = \begin{cases} 2 & \text{if } i = 2 \text{ and } \text{ i} \equiv 2\,(mod\ 3), (1 \leq i \leq \lceil \frac{n}{2} \rceil) \\ -1 & \text{otherwise} \end{cases}$$

(ii) $n \equiv x\,(mod\ 6)$, where $x \in \{1, 2\}$

$$g(v_i) = \begin{cases} 2 & \text{if } i = 2 \text{ and } \text{ i} \equiv 2\,(mod\ 3), (1 \leq i \leq \lfloor \frac{n}{2} \rfloor) \\ +1 & \text{if } i = n \\ -1 & \text{otherwise} \end{cases}$$

Then we can verify that $g(N[v]) \geq 0$ for at least half of the vertices in G with weight $\gamma^{NN}_{WMR}(P_n) \leq \begin{cases} 3 \lceil \frac{n}{6} \rceil - n - 1 & \text{if } \lceil \frac{n}{2} \rceil \equiv 1\,(mod\ 3), \\ 3 \lceil \frac{n}{6} \rceil - n & \text{otherwise.} \end{cases}$

Consequently, the result follows. □

Theorem 3.0.7. *For any cycle C_n, on $(n \geq 3)$ vertices,* $\gamma^{NN}_{WMR}(C_n) = 3 \lceil \frac{n}{6} \rceil - n$.

Proof. Let $C_n = (v_1, v_2, ..., v_n, v_1)$ and let f be a NNWMRDF of C_n with $\gamma^{NN}_{wmr}(C_n) = f(V)$. Since $\deg v_i = 2$, for all i, for a vertex $(v_i \in V_{f,2})$, three vertices belongs to N_f. Since $|N_f| \geq \lceil \frac{n}{2} \rceil$, we have $|V_{f,2}| \geq \lceil \frac{n}{6} \rceil$ and $|V_{f,-1}| \leq n - \lceil \frac{n}{6} \rceil$. Hence $\gamma^{NN}_{WMR}(C_n) \geq 3 \lceil \frac{n}{6} \rceil - n$.

On the other hand, define the function $g : V \rightarrow \{-1, +1, 2\}$ by

(i) $n \equiv x\,(mod\ 6)$, where $x \in \{1, 2\}$

$$g(v_i) = \begin{cases} 2 & \text{if } i = 2 \textit{ and } \text{ i} \equiv 2\,(mod\ 3), (2 \leq i \leq \lceil \frac{n}{2} \rceil + 1) \\ -1 & \text{otherwise} \end{cases}$$

(ii) $n \equiv x\,(mod\ 6)$, where $x \in \{0, 3, 4, 5\}$

$$g(v_i) = \begin{cases} 2 & \text{if } i = 2 \textit{ and } \text{ i} \equiv 2\,(mod\ 3), (2 \leq i \leq \lceil \frac{n}{2} \rceil) \\ -1 & \text{otherwise} \end{cases}$$

Then we can verify that $g(N[v]) \geq 0$ for at least half of the vertices in C_n with weight $\gamma^{NN}_{WMR}(C_n) \leq 3 \lceil \frac{n}{6} \rceil - n$ □

Theorem 3.0.8. *For any complete graph K_n, on $(n \geq 2)$ vertices, $\gamma^{NN}_{WMR}(K_n) = 0$*

Proof. Let f be a NNWMRDF of K_n with $\gamma^{NN}_{WMR}(K_n) = f(V)$. Then there exists an vertex v of K_n such that $f(N[v]) \geq 0$. This implies that $\gamma^{NN}_{WMR}(K_n) = f(V) = f(N[v]) \geq 0$.

On the other hand, Choose vertices $v_1, v_2, ..., v_n$ of K_n and let $S = \left\{v_1, v_2, ..., v_{\lceil \frac{n}{2} \rceil}\right\}$. Suppose first that n is even and $n \geq 4$. Now define $g : V \rightarrow \{-1, +1, 2\}$ by $g(x) = -1$, for each $x \in S$, and $g(x) = 1$ for each $x \notin S$. Now suppose n is odd. Define $g(v_n) = 2$, $g(x) = -1$, for each $x \in S$, and $g(x) = +1$, for each $x \notin S \cup \{v_n\}$. Hence $\gamma^{NN}_{WMR}(K_n) \leq 0$. Consequently the result follows. □

Theorem 3.0.9. *For any star $K_{1,n-1}$ on $n \geq 2$ vertices,*

$$\gamma_{WMR}^{NN}(K_{1,n-1}) = 2 - n.$$

Proof. Let u be the central vertex and let $v_1, v_2, ..., v_{n-1}$ be the pendant vertices. Now let f be a NNWMRDF of $K_{1,n-1}$ such that $\gamma_{WMR}^{NN}(K_{1,n-1}) = f(V)$. Since $K_{1,n-1}$ is connected, $|V_{f,1}| \geq 1$. Hence $\gamma_{WMR}^{NN}(K_{1,n-1}) \geq 2 - n$. On the other hand by assigning $+1$ to the central vertex and -1 for pendant vertices we obtain a NNWMRDF of $K_{1,n-1}$ with weight $2 - n$. Therefore $\gamma_{WMR}^{NN}(K_{1,n-1}) = 2 - n$. □

Theorem 3.0.10. *For any wheel W_n, on $(n \geq 4)$ vertices,*

$$\gamma_{WMR}^{NN}(W_n) = 2\left\lceil \frac{n}{6} \right\rceil - n + 2.$$

Proof. Let u be the central vertex of W_n and let $v_1, v_2, ..., v_{n-1}$ be the vertices of C_{n-1}. Now let f be a NNWMRDF of W_n such that $\gamma_{WMR}^{NN}(W_n) = f(V)$.If $u \in N_f$, then $\gamma_{WMR}^{NN}(W_n) \geq 0$. Now suppose $u \notin N_f$. For each $v_i \in N_f$, we have

(i) $f(N[v_i]) - f(u) \geq -1$ if $f(u) = 2$

(ii) $f(N[v_i]) - f(u) \geq -1$ if $f(u) = 1$

(iii) $f(N[v_i]) - f(u) \geq 1$ if $f(u) = -1$

Since f is minimum, we take $f(u) = +1$. Now since every vertex $v_i \in N_f$ has degree three, for a vertex $v_i \in V_{f,1}$, three vertices belongs to N_f. Also since $|N_f| \geq \left\lceil \frac{n}{2} \right\rceil$ and $f(u) = +1$, we have $|V_{f,1}| \geq \left\lceil \frac{n}{6} \right\rceil + 1$. Therefore $|V_{f,-1}| \leq n - \left\lceil \frac{n}{6} \right\rceil - 1$. Hence the lower bound follows.

Now for the upper bound define the function $g : V \to \{-1, +1, 2\}$ by $g(u) = +1$ and

$$g(v_i) = \begin{cases} 1 & \text{for } any \left\lceil \frac{n}{6} \right\rceil \ vertices\ with\ i \equiv 1(mod 3), \\ -1 & \text{otherwise.} \end{cases}$$

It is easy to verify that g is a NNWMRDF with weight $2\left\lceil \frac{n}{6} \right\rceil - n + 2$. Consequently the result follows.

Theorem 3.0.11. *Let G be a regular graph of order n. Then* $\gamma^{NN}_{WMR}(G) \geq \left\lceil \frac{n}{2} \right\rceil - n$.

Proof. Let f be a NNWMRDF of G. Since for at least half of the vertices $v \in V$, $f(N[v]) \geq 0$, we have

$$\sum_{v \in V} f(N[v]) \geq 0(\left\lceil \tfrac{n}{2} \right\rceil) + (-r-1)(n - \left\lceil \tfrac{n}{2} \right\rceil)$$

Also the $\sum_{v \in V} f(N[v])$ counts the value $f(v)$ exactly $deg\ v + 1$ times for each vertex $v \in V$. That is, $\sum_{v \in V} f(N[v]) = \sum_{v \in V} f(v)(deg\ v + 1)$. Hence,

$$\sum_{v \in V} f(v)(r+1) \geq (0)(\left\lceil \tfrac{n}{2} \right\rceil) + (-r-1)(n - \left\lceil \tfrac{n}{2} \right\rceil)$$

$$= \left\lceil \tfrac{n}{2} \right\rceil (r+1) + n(-1-r)$$

$$f(V) \geq \frac{\left\lceil \frac{n}{2} \right\rceil (r+1) - n(1+r)}{r+1}$$

□

Theorem 3.0.12. *Let$G$=$K_{2n} - M$; where M is a perfect matching in the complete graph K_{2n}. Then $\gamma^{NN}_{WMR}(G) = -1$.*

Proof. Let $V(K_{2n}) = (v_1, v_2, ..., v_{2n})$ and $M = (v_1v_2, v_3v_4, v_5v_6, ..., v_{2n-1}v_{2n})$. Now define a function $f : V \rightarrow \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} +1 & \text{if } 1 \leq i \leq n-2 \\ 2 & \text{if } i = n \\ -1 & \text{otherwise} \end{cases}$$

Then it is easy to verify that f is a NNWMRDF. Hence $\gamma_{WMR}^{NN}(G) \leq -1$. Now let g be any NNWMRDF of G with $\gamma_{WMR}^{NN}(G) = g(V)$. Let $v_i \in N_g$ and let v_j be the vertex which is not adjacent to v_i. Hence $g(V) = g(N[v_i]) + g(v_j) \geq 0 - 1 = -1$. □

Theorem 3.0.13. *Let G denote the friendship graph with t-triangles. Then*

$$\gamma_{WMR}^{NN}(G) = 2 - 2t$$

Proof. Let u be the central vertex of G and let $(v_1, v_2, ..., v_{2t})$ be the vertices in the triangles. Now define $f : V \to \{-1, 1, 2\}$ by $f(u) = 2$ and assign -1 for remaining vertices. Then it is easy to verify that f is a NNWMRDF with $\gamma_{WMR}^{NN}(G) \leq 2 - 2t$

Now let g be any NNWMRDF with $\gamma_{WMR}^{NN}(G) = g(V) = g(N[v_i]) + g(V - N[v_i])$. Therefore $\gamma_{WMR}^{NN}(G) \geq 0 + (-1)[2(t-1)]$. □

4 Open problems

We encountered numerous problems in the course of this investigation. We list here some of them.

1. Characterization of graphs G for which $\gamma_{wsR}(G) = \gamma_{WMR}^{NN}(G)$.
2. Find sharp lower bound of $\gamma_{WMR}^{NN}(G)$.
3. Find the non-negative weak majority roman domination number of trees.

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